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Aug 2013

Discretisiation of Relaxation Time Spectrum

→ Rheol Acta 28:511 (1989)
→ J Non Newt Fluid Dyn 68:225 (1997)

Definition of Spectrum $H(\tau)$ from Relaxation Modulus

(1) $$G(t) = \int_0^{\tau_{max}} \frac{d\tau}{\tau} H(\tau) e^{-t/\tau}$$

Discrete relaxation modulus

(2) $$G(t) = \sum_{i=1}^{N} g_i e^{-t/\tau_i}$$

Splitting of integral (eq. 1)

(3) $$G(t) = \int_0^{\tau_1} + \int_{\tau_1}^{\tau_2} + \cdots \int_{\tau_i}^{\tau_{i+1}} + \cdots \int_{\tau_{N-1}}^{\tau_N}$$

Compare eq.s 2 and 3

$$(4) \quad g_i e^{-t/\tau_i} = \int_{\tau_{i-1}}^{\tau_i} \frac{d\tau}{\tau} H(\tau) e^{-t/\tau}$$

A better interpolation is achieved when centralizing the time interval around τ_i

$$(5) \quad g_i e^{-t/\tau_i} = \int_{\tau_i^-}^{\tau_i^+} \frac{d\tau}{\tau} H(\tau) e^{-t/\tau}$$

with $\tau_i^+ = \sqrt{\tau_i \tau_{i+1}}$ see Fig 1 1992 paper

$\tau_i^- = \sqrt{\tau_i \tau_{i-1}}$

since plotting occurs on a log scale in time. For a sufficiently small interval, $H(\tau) e^{-t/\tau}$ is nearly constant

$$(6) \quad H(\tau) e^{-t/\tau} \approx H(\tau_i) e^{-t/\tau_i} = \text{const}$$

for $\tau_i^- < \tau < \tau_i^+$

This simplifies the equation to

$$(7) \quad g_i e^{-t/\tau_i} \approx H(\tau_i) e^{-t/\tau_i} \int_{\tau_i^-}^{\tau_i^+} \frac{d\tau}{\tau}$$

or

$$g_i \approx H(\tau_i) \int_{\tau_i^-}^{\tau_i^+} \frac{d\tau}{\tau}$$

$$\ln \frac{\tau_i^+}{\tau_i^-} = \ln \sqrt{\frac{\tau_{i+1}\tau_i}{\tau_{i-1}\tau_i}} = \ln \sqrt{\frac{\tau_{i+1}}{\tau_{i-1}}}$$

Result

$$(8) \quad \boxed{g_i = H(\tau_i) \ln \sqrt{\frac{\tau_{i+1}}{\tau_{i-1}}}}$$

eq. 9 in 1992 paper

Convention:
τ-value decreases with increasing i

$$(9) \quad g_i = H(\tau_i) \ln \sqrt{\frac{\tau_{i-1}}{\tau_{i+1}}}$$

Possible variation:
include $e^{-t/\tau}$ inside the integral

Variation of equations 7 and 4

$$(10) \quad g_i e^{-t/\tau_i} = H(\tau_i) \int_{\tau_i^-}^{\tau_i^+} \frac{d\tau}{\tau} e^{-t/\tau}$$

substitute $x = t/\tau$

$$dx = -\frac{t}{\tau^2} d\tau \Rightarrow \frac{d\tau}{\tau} = -\frac{dx}{x}$$

$$(11) \quad I = \int \frac{d\tau}{\tau} e^{-t/\tau} = -\int \frac{dx}{x} e^{-x}$$

$$= \ln|x| + \frac{-x}{1} + \frac{x^2}{2 \cdot 2!} - \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \cdot\cdot + \frac{(-1)^n x^n}{n \cdot n!} + \ldots$$

I do not know how much improvement we will get from this more accurate representation.

Furthermore, we could iterate by getting a first solution of $H(\tau)$ and determine the slope $(\partial H/\partial \tau)_{\tau_i}$. This will allow a Taylor expansion of $H(\tau)$ near τ_i. That expansion could also be included in an improved solution of eq. 4.

Notes

- g_i belongs to a discrete mode with time constant τ_i

- $H(\tau_i)$ is a point on the continuous relaxation spectrum

$$H(\tau_i) = \left(H(\tau)\right)_{\tau=\tau_i}$$

- equation 5 does not apply to the end point ($i = 1$). Extra assumptions need to be introduced.