Discretization of Relaxation Time

Spectrum


Definition of Spectrum $H(t)$ from Relaxation Modulus $G(t)$:

$$G(t) = \int_0^{T_{\text{max}}} d\tau \; H(\tau) e^{-t/\tau}$$

(1)

Discrete relaxation modulus

$$G(t) = \sum_{i=1}^{N} q_i e^{-t/\tau_i}$$

(2)

Splitting of integral (eq. 1)

$$G(t) = \int_0^{\tau_1} + \int_{\tau_1}^{\tau_2} + \int_{\tau_2}^{\tau_3} + \ldots + \int_{\tau_{N-1}}^{\tau_N}$$

(3)
Compare eqs 2 and 3

\[ g_i e = \int_{\tau_i^{-}}^{\tau_i^{+}} \frac{dt}{\tau} \quad \text{for} \quad -t/\tau_i \leq t/\tau_i \leq t_i \]

A better interpolation is achieved when centralizing the time interval around \( \tau_i \):

\[ g_i e = \int_{\tau_i^{-}}^{\tau_i^{+}} \frac{dt}{\tau} \quad \text{for} \quad -t/\tau_i \leq t/\tau_i \leq t_i \]

With \( \tau_i^{+} = \sqrt{\tau_i^{-} \tau_i^{+1}} \) \quad \text{and} \quad \tau_i^{-} = \sqrt{\tau_i^{-1} \tau_i^{-1}} \]

Note Fig 1

1992 paper

Since plotting occurs on a log scale in time, the function \( H(t) \) is nearly constant

\[ H(t) e^t \approx H(t_i) e^t = \text{const} \]

for \( \tau_i^{-} < t < \tau_i^{+} \)

\[ -2 \]
This simplifies the equation to

\[ g_i e^{-t/t_i} \approx H(t_i) e^{-t/t_i} \int_{t_i}^{t_i^+} \frac{dz}{z} \]

or

\[ g_i \approx H(t_i) \int_{t_i^-}^{t_i^+} \frac{dz}{z} \]

\[ \ln \frac{t_i^+}{t_i^-} = \ln \sqrt{\frac{t_i}{t_i^+}} = \ln \sqrt{\frac{t_i}{t_i^-}} \]

**Result**

\[ g_i = H(t_i)_\ln \sqrt{\frac{t_i}{t_i^-}} \]

Eq. 9 in 1982 paper

**Convention:**

T-value decreases with increasing i

\[ g_i = H(t_i)_\ln \sqrt{\frac{t_i}{t_i^-}} \]

Possible variation:

Include e^{-t/t_i} inside the integral
Variation of equations 7 and 4

(10) \[ g_i e^{-t/t_i} = H(t_i) \int_{t_i}^{t_i+T} \frac{d\tau}{\tau_i} e^{-\tau/t_i} \]

Substitute \( x = t/t_i \)

\[ dx = -\frac{x}{t_i} dt \Rightarrow \frac{d\tau}{\tau_i} = -\frac{dx}{x} \]

(11) \[ I = \int \frac{d\tau}{\tau} e^{-t/\tau} = - \int \frac{dx}{x} e^{-x} \]

\[ = \ln |x| + \frac{x}{1} + \frac{x^2}{2 \cdot 2!} - \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \cdots + \frac{(-1)^n x^n}{n \cdot n!} + \cdots \]

I do not know how much this improvement we will get from this more accurate representation.

Furthermore, we could iterate by getting a first solution of \( H(t) \) and determine the slope \( \left( \partial H / \partial t \right)_{t_i} \). This will allow a Taylor expansion of \( H(t) \) near \( t_i \). That expansion could also be included in our improved solution of eq. 7.
Notes

- \( g_i \) belongs to a discrete mode with time constant \( \tau_i \)

- \( H(\tau_i) \) is a point on the continuous relaxation spectrum

\[
H(\tau_i) = \left( \frac{H(\infty)}{\tau_i} \right) ^{\frac{1}{\tau_i}}
\]

- Equation 5 does not apply to the end point \( (i = 1) \). Extra assumptions need to be introduced.