Viscous Dissipation Term in Energy Equations

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OBJECTIVES

After completing this module, the student should be able to:

1. Calculate the rate of viscous dissipation for a given flow.
2. Calculate the rate of viscous dissipation in a macroscopic energy balance.
3. Calculate dissipation in a slip layer.
4. Use the dissipation function as a criterion to distinguish between a viscous and an elastic material.
5. Calculate dimensionless groups to estimate the magnitude of viscous dissipation.

PREREQUISITE MATHEMATICAL SKILLS

1. First year college calculus.

PREREQUISITE ENGINEERING AND SCIENCE SKILLS

1. First year college physics.
2. Macroscopic balances.

INTRODUCTION

Deformation and flow of materials require energy. This mechanical energy is dissipated, i.e. during the flow it is converted into internal energy (heat) of the material. This phenomenon can be demonstrated by performing a simple experiment with a metal paper clip: bend the clip wide open and close it repeatedly until the clip breaks. Now, touch the metal near the region of the break and feel the high temperature. The mechanical energy for bending the metal has been converted into internal energy. The increase of internal energy expresses itself in a temperature rise.

Viscous dissipation is of interest for many applications: significant temperature rises are observed in *polymer processing* flows such as injection molding or extrusion at high rates. Aerodynamic heating in the thin *boundary layer* around high speed aircraft raises the temperature of the skin. In a completely different application, the dissipation function is used to define the *viscosity of dilute suspensions* (Einstein, 1906, 1911): Viscous dissipation for a fluid with suspended particles is equated to the viscous dissipation in a pure Newtonian fluid, both being in the same flow (same macroscopic velocity gradient).

RATE OF VISCOUS DISSIPATION

The rate at which work is being done on a volume element for changing its volume and its shape is defined as (for derivation, see Appendix)

$$\sigma : \nabla v = \underbrace{-p\nabla \cdot v} + \underbrace{\tau : \nabla v} \tag{1}$$

| rate of work for volume change | rate of work for shape change at constant volume |

The stress,

$$\sigma = -p\mathbf{1} + \tau \tag{2}$$

with

$$p = -\frac{1}{3} \text{ trace } \sigma$$

is divided into the pressure, p, and the extra stress, τ. $\nabla \cdot v$ and ∇v are the divergence of the velocity vector and the velocity gradient. The second term in Equation 1 is called the "dissipation function,"

$$\phi = \tau : \nabla v \tag{3}$$

since most (not necessarily all) of the work is irreversibly converted into heat. The dissipation function for flows of Newtonian fluids is given in component form; see Table 1.

VISCOUS DISSIPATION IN PIPE FLOW

The steady flow in a pipe of constant cross section (radius R) will be used in the following for explaining viscous dissipation in bulk and at a slip boundary.

Macroscopic Balance

Flow of a fluid in a pipe requires mechanical energy which is supplied by a pump or by the hydrostatic pres-

Cartesian Coordinates

$$\tau : \nabla \mathbf{v} = 2\mu \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 - \frac{1}{3} (\nabla \cdot \mathbf{v})^2 \right]$$

$$+ \mu \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right]^2 + \mu \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right]^2 + \mu \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right]^2$$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Cylindrical Coordinates

$$\tau : \nabla \mathbf{v} = 2\mu \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 - \frac{1}{3} (\nabla \cdot \mathbf{v})^2 \right]$$

$$+ \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2 + \mu \left[\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right]^2 + \mu \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right]^2$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

Spherical Coordinates

$$\tau : \nabla \mathbf{v} = 2\mu \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right)^2 - \frac{1}{3} (\nabla \cdot \mathbf{v})^2 \right]$$

$$+ \mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2 + \mu \left[\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right]^2$$

$$+ \mu \left[\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \right]^2$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

sure of a reservoir (potential energy). Consider a pipe segment as shown in Figure 1 and a control volume between cross-sections 1 and 2.

The rate of work done for flow of a fluid through a pipe is calculated by integrating the rate of work per unit surface area, $\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{v}$, over the entire surface of the control volume. See also the Appendix Equation 46. Note that the surface along the pipe wall does not contribute, since its velocity is zero. The work on cross-sections 1 and 2 can be calculated by assuming uniform pressures p_1 and p_2 and by neglecting the small influence of the extra stress τ. The rate of work done on the volume of fluid becomes

$$E = \int_0^R p_1 v_1 2\pi r \, dr - \int_0^R p_2 v_2 2\pi r \, dr \tag{4}$$

The volume flow rate is the same in both cross sections (assuming constant density)

$$Q = 2\pi \int_0^R v_1 r \, dr = 2\pi \int_0^R v_2 r \, dr \tag{5}$$

where ρ = const. This gives a total rate of work

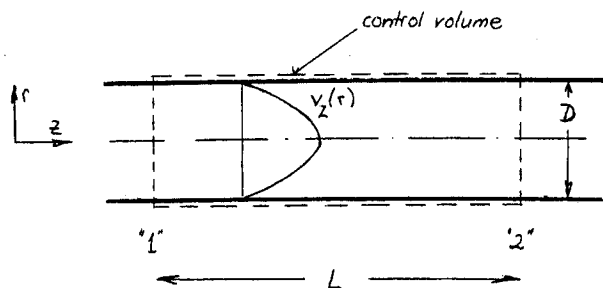$$E = Q(p_1 - p_2) \tag{6}$$

Figure 1. Straight pipe section of length L, diameter D. The control volume includes a thin layer of the *stationary* wall material.

for deforming, for accelerating, and for elevating the fluid. In a horizontal pipe of uniform cross-section (Figure 1), the mechanical energy for the pipe flow is completely dissipated, since the kinetic energy (no acceleration) and the potential energy (no change in altitude) of the fluid do not change between cross-sections 1 and 2. Note that the macroscopic energy balance gives the total dissipated energy ("friction loss"), however, no information is given on whether the dissipation is uniform throughout the volume or whether there are regions of large viscous dissipation and other regions of negligible dissipation. An interesting situation arises when the fluid

slips at the wall (see Example 3): one part of the energy for flow through the pipe is dissipated at the slip surface and the remaining part is dissipated in the volume of the deforming fluid.

Example 1: Pipe Flow in Polymer Processing

Polymer melt is forced through a pipe of $L = 0.04$ m and $D = 0.002$ m (runner channel to fill the mold of injection molding machine). A pressure drop $p_1 - p_2 = 8 \times 10^7$ Pa was typically found to give a volume flow rate $Q = 6 \times 10^{-6}$ m^3/s. Calculate the average temperature increase in the polymer between inlet and outlet. Assume adiabatic walls, negligible density changes, and a steady temperature field in the pipe.

Typical values for the physical properties of a polymer are:

density $\rho = 10^3$ kg/m^3

heat capacity $c = 1.4 \times 10^3$ J kg^{-1} K^{-1}

The dissipated energy is calculated from Equation 6:

$$E = Q(p_1 - p_2) = (6 \times 10^{-6})\,(8 \times 10^7) = 4.8 \times 10^2 \text{ J/s} \tag{7}$$

For a steady temperature field in a pipe with adiabatic walls, the entire energy is transported convectively with the fluid. The convective energy flow through a pipe cross section is

$$\rho c \langle T \rangle Q = 2\pi \int_0^R \rho c T(r) v(r) r \, dr \tag{8}$$

The difference between energy convection into the pipe and out of the pipe is equal to the generation of internal energy due to viscous dissipation

$$\rho c Q(\langle T \rangle_2 - \langle T \rangle_1) = Q(p_1 - p_2) \tag{9}$$

The average adiabatic temperature increase between inlet and outlet is calculated as

$$\Delta \langle T \rangle = \langle T \rangle_2 - \langle T \rangle_1 = \frac{p_1 - p_2}{\rho c}$$

$$= \frac{8 \times 10^7}{10^3\,(1.4 \times 10^3)} = 57 \text{ K} \tag{10}$$

Local temperatures might by far exceed this average value. An average temperature increase of 57 K is very large. Some of the assumptions in the beginning of this example will have to be reconsidered: the thermal and the rheological properties can be expected to change significantly between entrance and exit; the high temperatures of the fluid give rise to a substantial radial temperature gradient, i.e. the heat flux into the wall cannot be neglected anymore.

Example 2: Friction Loss in Pipe Flow of Water

Water at 20°C is flowing through a pipe ($L = 30.5$ m, $D = 0.05$ m) at an average velocity of 6.2 m/s. The pressure drop was found to be $\Delta p = 2.34 \times 10^5$ Pa. Calculate the temperature increase between inlet and outlet by assuming no conduction through the wall. The relevant properties are:

density $\rho = 1000$ kg m^{-3}

heat capacity $c_p = 4.2 \times 10^3$ J kg^{-1} K^{-1}

The average temperature increase for adiabatic pipe flow is calculated as

$$\Delta \langle T \rangle = \frac{\Delta p}{\rho c_p} = \frac{2.34 \times 10^5}{10^3\,(4.2 \times 10^3)} = 5.6 \times 10^{-2} \text{ K} \tag{11}$$

Viscous dissipation does *not* significantly alter the temperature of the water. However, it is still important, since it determines the power requirement of a pipe line system, i.e. viscous dissipation determines the size of the pumps for a pipe system and the energy costs of pumping.

Example 3: Pipe Flow With Slip at Wall

Consider a fluid which flows through a straight pipe section as shown in Figure 1. The fluid is found to slip at the wall at a velocity v_R. Examples of slipping fluids are highly filled suspensions, linear polyethylene, polybutadiene, and polyvinylchloride in the molten state. Determine how much energy is dissipated in the slip region.

A control volume for a macroscopic energy balance is chosen around the fluid volume in the pipe section. The main point is that the control volume does *not* contain a layer of stationary wall material, but that the outer surface moves with the finite slip velocity v_R. The rate of work done on the control volume of fluid is again calculated by integrating the rate of work per unit surface, $\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{v}$, over the entire surface.

$$E_d = Q(p_1 - p_2) + \int_0^L (\sigma_{rz} V)_R \pi D \, dz \tag{12}$$

The first part is the rate of work done for flow through the pipe (see Equation 6) and the second part is (minus) the rate of work for slip along the pipe wall. The shear stress at the wall can be expressed in terms of the axial pressure gradient in the pipe,

$$(\sigma_{rz})_R = \frac{R}{2} \frac{\partial p}{\partial z} \tag{13}$$

In case of a uniform slip velocity v_R and a uniform pressure gradient, $\partial p / \partial z = (p_2 - p_1)/L$, the integral can be simplified. The rate of work for slip along the wall becomes

$$E_s = -\pi D \int_0^L (\sigma_{rz} v)_R \, dz = (p_1 - p_2) v_R \pi D^2/4 \tag{14}$$

The rate of work for deforming the fluid (subscript d) in

the flow through the pipe remains as

$$E_d = Q(p_1 - p_2) - E_s$$

$$= (p_1 - p_2)Q \left\{ 1 - \frac{v_R}{\langle v \rangle} \right\} \tag{15}$$

$\langle v \rangle$ is the average velocity. The limiting case of plug flow, $v_R = \langle v \rangle$, obviously requires no energy for the deformation. The other limiting case of no slip, $v_R \ll \langle v \rangle$, requires all the energy to be dissipated in the deforming fluid.

DISTRIBUTION OF THE DISSIPATION FUNCTION

Most flows are inhomogeneous, i.e. the stress and the rate of deformation are functions of position. Again, steady pipe flow is used for demonstrating inhomogeneity.

In pipe flow, the viscous dissipation is not uniform in the cross-section (see Table 1)

$$\phi = \tau_{rz} \dot{\gamma}_{rz} \tag{16}$$

The shear stress is given by the stress equation of motion,

$$\tau_{rz} = \frac{r}{2} \frac{dp}{dz} \tag{17}$$

where

$$\frac{dp}{dz} = \text{axial pressure gradient}$$

The shear rate is equal to the radial velocity gradient

$$\dot{\gamma}_{rz} = \frac{\partial v_z}{\partial r} \tag{18}$$

which depends on the type of fluid:

Newtonian fluid: $\dot{\gamma}_{rz} = -4 \dfrac{\langle v_z \rangle}{R} \dfrac{r}{R}$ (19)

Power law fluid: $\dot{\gamma}_{rz} = -\left(\dfrac{1}{n} + 3 \right) \dfrac{\langle v_z \rangle}{R} \left(\dfrac{r}{R} \right)^{1/n}$ (20)

The power law viscosity is here defined by

$$\eta(\dot{\gamma}) = \eta^o |\dot{\gamma}/\dot{\gamma}^o|^{n-1}$$

in

$$\tau_{rz} = \eta(\dot{\gamma}) \, \dot{\gamma}_{rz} \tag{21}$$

with a reference viscosity of $\eta^o = \eta(\dot{\gamma}^o)$ at a reference shear rate $\dot{\gamma}^o$ in the shear thinning region of the viscosity plot.

Now, let us determine the distribution of the dissipation function for the power law fluid, keeping in mind that the solution will include the Newtonian fluid as a special case (with $n = 1$). Introducing Equations 20 and
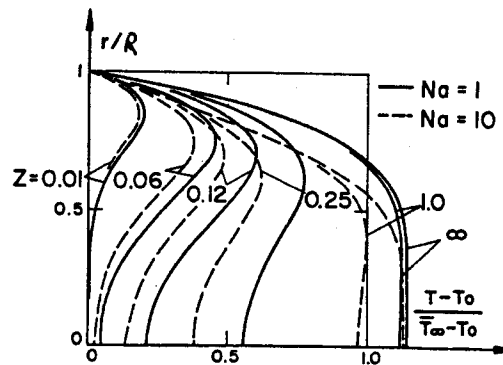

Figure 2. Developing temperature profiles in a pipe with isothermal wall (14). Parameters are the Nahme number, Na, and the dimensionless distance from the entrance $Z = k/(\rho c_p \langle v \rangle R^2)$. The viscosity is described by a power law with $n = 0.4$.

21 into the dissipation function ϕ, Equation 16, gives

$$\phi(r) = \eta^o \left(\frac{1}{n} + 3 \right)^{n+1} \left(\frac{\langle v_z \rangle}{R} \right)^2 \left(\frac{r}{R} \right)^{1 + 1/n} \tag{22}$$

where

$$\eta^o = \eta \left(\frac{\langle v_z \rangle}{R} \right) \; ; \; \dot{\gamma}^o = \langle v_z \rangle / R$$

The dissipation function is zero at the center line and has its maximum value at the wall.

Developing temperature fields in pipe flow demonstrates the non-uniformity of the viscous dissipation; see Figure 2. Large radial temperature differences are generated in a fluid which started out being of uniform temperature. These radial temperature differences give rise to conduction of heat towards the wall. A fully developed temperature is reached when the heat flow into the wall balances the viscous heat generation. A method of calculating developing temperatures in many different shear flow geometries and a review of the literature are given by Winter (15).

DIMENSIONLESS GROUPS

The calculation of temperature fields requires the solution of the equation of energy. The equation of energy is conveniently rewritten in dimensionless form. Order of magnitude arguments allow the elimination of small terms, as compared to the important terms. This procedure also applies for the viscous dissipation term in the energy equation. One has to define a dimensionless group, the generation number N_{Gn}, which indicates the importance of viscous dissipation as compared to convection, conduction, and compression.

The generation number N_{Gn} can be defined with the equation of energy in a most simple form:

$$\rho c_p \frac{DT}{Dt} = k \nabla^2 T + \frac{1}{2} \eta \dot{\gamma} : \dot{\gamma} \tag{23}$$

The fluid is assumed to be purely viscous and of constant viscosity. The thermal properties ρ, c_p, k are assumed to

be constant [an equivalent discussion for the compressible fluid can be found in Schlichting (13)]. The stress in the viscous dissipation term is already replaced by

$$\tau = \eta \dot{\gamma} \text{ with } \dot{\gamma} = \nabla v + (\nabla v)^T \qquad (24)$$

The viscosity has to be specified as a function of temperature and deformation rate. D/Dt is a substantial time derivative.

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v \cdot \nabla T \qquad (25)$$

The equation of energy is made dimensionless by scaling it with the factor $H^2/k\Delta T^o$. Equation 23 becomes

$$\underbrace{\frac{\rho c_p V H}{k} \frac{H}{L} \frac{DT^*}{Dt^*}}_{\text{Graetz number, } N_{Gz}}$$

$$= \nabla^* T^* + \frac{1}{2} \underbrace{\frac{V^2 \eta^o}{k\Delta T^o}}_{\text{Generation number, } N_{Gn}} \eta^* (\dot{\gamma}^* : \dot{\gamma}^*) \qquad (26)$$

The scaling factors,

V = reference velocity
H = characteristic length in direction of velocity gradient (pipe radius or slit width, for instance)
L = characteristic length in flow direction (pipe length in pipe flow)
ΔT^o = reference temperature difference
T_o = reference temperature
η^o = reference viscosity, $\eta(V/H, T_o)$

make the variables dimensionless:

T^* = $(T - T_o)/\Delta T^o$
t^* = tL/V
∇^* = $H\nabla$
η^* = η/η^o
$\dot{\gamma}^*$ = $\dot{\gamma}H/V$

The definition of the generation number is

$$N_{Gn} = \frac{V^2 \eta^o}{k\Delta T^o} \qquad (27)$$

and its relation to the equation of energy is known. A large generation number implies that viscous dissipation cannot be neglected in comparison with heat conduction. Note that the product $\eta^* \dot{\gamma}^* : \dot{\gamma}^*$ might locally adopt very large values ($\gg 1$). Therefore, the viscous dissipation might locally influence the temperature even if the generation number is smaller than one. A safe value for neglecting the effects of viscous dissipation seems to be $N_{Gn} \ll 0.1$.

The generation number has been defined several ways and accordingly, has different names in the literature. This module will discuss two of the most common cases. The other dimensionless group, the Graetz number, compares the magnitude of convection and conduction. It will not be discussed here.

Flow With a Transverse Temperature Difference

There are many flows with a given temperature difference, $\Delta T_{process}$, between the fluid and a wall,

$$\Delta T_{process} = T_f - T_w \qquad (28)$$

or between the boundaries of the flow,

$$\Delta T_{process} = T_{w1} - T_{w2} \qquad (29)$$

Then this temperature difference is chosen to be the characteristic temperature difference, ΔT^o, with which the temperature changes can be scaled:

$$\Delta T^o = \Delta T_{process} \qquad (30)$$

The generation number in this case is called the Brinkman number (4),

$$N_{Br} = \frac{V^2 \eta^o}{k\Delta T_{process}}$$

or the product of the Prandtl number, N_{Pr}, and the Eckert number, N_{Ec} (13),

$$N_{Ec} \cdot N_{Pr} = \frac{V^2}{c_p \Delta T_{process}} \cdot \frac{c_p \eta^o}{k} = N_{Br}$$

Both definitions are equivalent. The name, "Brinkman number," seems to be preferred in studies on developing temperatures in channel flow and the name, "Eckert number," is preferably used in studies on viscous dissipation in thermal boundary layers.

Flow Without Imposed Temperature Difference

Scaling of the temperature changes due to viscous dissipation becomes more difficult when there is no given temperature difference $\Delta T_{process}$. This situation always occurs when isothermal flow conditions are attempted (which is quite common in polymer processing). Viscous dissipation would disturb these isothermal conditions and its extent has to be estimated in modeling efforts.

The most common choice of scaling factor for temperature changes in nearly isothermal processes is

$$\Delta T^o = \Delta T_{rheol} = -\left(\frac{\eta}{\partial \eta / \partial T}\right)_{T_o, \dot{\gamma}^o} \qquad (33)$$

The temperature changes are of interest since they affect the viscosity and hence, affect the flow pattern. A viscosity with a temperature dependence of the Arrhenius type,

$$\eta = ae^{b/T} \qquad (34)$$

where T = absolute temperature

gives a characteristic rheological temperature difference,

$$\Delta T_{rheol} = \frac{T_o^2}{b}$$

where T_o = absolute reference temperature

For molten polymers, $\Delta T_{rheol} = 30\text{-}70$ K

The temperature differences are then scaled as $(T - T_o)/\Delta T_{\text{rheol}}$ and the generation number is called the Nahme number (11).

$$N_{Na} = \frac{V^2 \eta^o}{k \Delta T_{\text{rheol}}} \qquad (36)$$

In nearly isothermal processes, the Nahme number is a measure of how much viscous dissipation affects the temperature dependent viscosity. Large values of N_{Na} indicate that isothermal conditions cannot be maintained.

Note that the temperature change and the generation number *cannot* be scaled with a temperature *level* T_o. For scaling, one always has to use a temperature *difference*. A quantity $(T - T_o)/T_o$ would be dimensionless; however, it would adopt different values in different temperature scales (Fahrenheit, Celsius, Kelvin).

ENERGY DISSIPATION AND STORAGE

The work done for deforming a *viscous* material is irreversibly converted into internal energy, i.e. it is dissipated. The work done for deforming an *elastic* material, however, is stored as potential energy, which can be recovered mechanically. An example is a rubber band in the stretched state. It can perform work when contracting to its original length. The dissipation function, $\tau : \nabla v$, is applicable to both cases. It gives the rate of work done for deforming a material, independent of whether this material is viscous, elastic, or viscoelastic. The dissipation function is always positive when applied to viscous materials. It adopts positive or negative values with elastic and with viscoelastic materials. The name "dissipation function" is actually misleading when describing storage and recovery phenomena in deforming elastic materials.

Example 4: Oscillatory Strain of a Hookean Material

An elastic material is placed between two extensive parallel plates, as shown in Figure 3. The lower plate is stationary and the upper plate moves with velocity.

$$v_h = v_o \cos (\omega t) \qquad (37)$$

at $y = h$

A momentum balance gives the velocity in between the plates. For uniform properties, it has the simple form

$$v_x(y) = \frac{y}{h} v_o \cos (\omega t) \qquad (38)$$

Calculate the rate of work done in order to deform the material. Integrate to find the total work for one cycle. Compare the result to the behavior of a Newtonian fluid in the same experiment.

The rate of work done is given by the dissipation function,

$$\tau : \nabla v = \tau_{xy} \dot\gamma_{xy} \qquad (39)$$

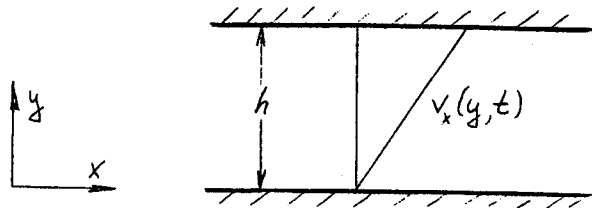The shear rate is given by the velocity field of the exper-

Figure 3. Sandwich device for shearing a material.

iment, Equation 38:

$$\dot\gamma_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} = \frac{v_o}{h} \cos (\omega t) \qquad (40)$$

The stress in the Hookean material is proportional to the shear strain,

$$\tau_{xy} = G \gamma_{xy} \qquad (41)$$

The dissipation function becomes

$$\tau_{xy} \dot\gamma_{xy} = G \gamma_{xy} \dot\gamma_{xy} = \frac{G}{\omega} \left(\frac{v_o}{h}\right)^2 \sin (\omega t) \cos (\omega t)$$

$$= \frac{G}{2\omega} \left(\frac{v_o}{h}\right)^2 \sin (2\omega t) \qquad (42)$$

One cycle requires work

$$W = \int_0^{2\pi/\omega} \tau_{xy} \dot\gamma_{xy} \, dt = \frac{G}{2\omega} \left(\frac{v_o}{h}\right)^2 \int_0^{2\pi/\omega} \sin (2\omega t) \, dt$$

$$= \frac{G}{2\omega} \left(\frac{v_o}{h}\right)^2 \left(\frac{-1}{2\omega}\right) \cos (2\omega t)\big|_0^{2\pi/\omega} = 0 \qquad (43)$$

As expected for the elastic material, the work per cycle is equal to zero. The work done in one half of a cycle is recovered in the other half.

The same experiment, however, with a Newtonian fluid, would be described by a dissipation function:

$$\tau_{xy} \dot\gamma_{xy} = \mu(\dot\gamma_{xy})^2 = \mu \left(\frac{v_o}{h}\right)^2 \cos^2 (\omega t) \qquad (44)$$

and work per cycle

$$W = \int_0^{2\pi/\omega} \mu \left(\frac{v_o}{h}\right)^2 \cos^2 (\omega t) \, dt$$

$$= \mu \left(\frac{v_o}{h}\right)^2 \pi \qquad (45)$$

The work is dissipated in the material.

APPENDIX: DERIVATION OF THE DISSIPATION TERM

Consider a small volume element (volume V, surface S) of a material which is deformed by a stress σ on the surface. The rate of work done on the surface of the material element is calculated by integrating $(n \cdot \sigma \cdot v)_S$ over

the entire surface,

$$E = \iint_S (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{v})_S \, dS \qquad (46)$$

$\mathbf{n}$ is the outward unit normal on the surface and $(\mathbf{n} \cdot \boldsymbol{\sigma})_S$ is the force per unit area of surface. Multiplication with the surface velocity gives the rate of work per unit surface.

The work done on the surface is used for deforming the material element (change of shape and volume), accelerating it, and increasing its potential energy. This is shown in the following. It will lead to an expression for the rate at which work is being done for deforming the volume element.

The surface integral of Equation 46 is replaced by a volume integral (using the Gauss theorem):

$$\iint_S (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{v})_S \, dS = \iiint_V \nabla \cdot \boldsymbol{\sigma} \cdot \mathbf{v} \, dV \qquad (47)$$

and the kernel in volume integral is replaced by the identity

$$\nabla \cdot \boldsymbol{\sigma} \cdot \mathbf{v} = \boldsymbol{\sigma} : \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \cdot \boldsymbol{\sigma} \qquad (48)$$

The physical meaning of the product $\mathbf{v} \cdot \nabla \cdot \boldsymbol{\sigma}$ can be explained from the stress equation of motion. Scalar multiplication of the local velocity $\mathbf{v}$ with the stress equation of motion (3),

$$\frac{\rho}{2} \frac{D}{Dt} v^2 = \mathbf{v} \cdot \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{v} \cdot \mathbf{g} \qquad (49)$$

gives an expression for $\mathbf{v} \cdot \nabla \cdot \boldsymbol{\sigma}$. D/Dt is the substantial time derivative. The last two equations are introduced into the volume integral, Equation 47. The result of the derivation is

$$\iint_S (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{v})_S \, dS$$

$$= \iiint_V \left[\boldsymbol{\sigma} : \nabla \mathbf{v} + \frac{\rho}{2} \frac{D}{Dt} v^2 - \rho \mathbf{v} \cdot \mathbf{g} \right] dV \qquad (50)$$

- Rate of change of potential energy per unit volume.
- Rate of change of kinetic energy per unit volume.
- Rate of work for changing the volume and shape, per unit volume (see Eq. 1).

$\boldsymbol{\sigma} : \nabla \mathbf{v}$ is the dissipation term in the equation of energy. For many applications in polymer processing, the changes in kinetic and potential energy are negligibly small; all the work done on the surface is practically used for deforming the volume element.

LITERATURE CITED

1. Armstrong, R. C., and H. H. Winter, "Heat Transfer for Non-Newtonian Fluids," in "Heat Exchanges Design and Data Book," Section 2.5.12, E. U. Schlünder Ed., Hemisphere Publ. London (1982).
2. Astarita, G., and G. Marrucci, "Principles of Non-Newtonian Fluid Mechanics," McGraw Hill, London (1974).
3. Bird, R. B., Stewart, W. E., and E. N. Lightfoot, "Transport Phenomena," Wiley, New York (1960).
4. Brinkman, H. C., Appl. Sci. Research, A2, 120–124 (1951).
5. Cox, H. W., and C. W. Macosko, AIChE J., 20, 785–795 (1974).
6. Dinh, S. M., and R. C. Armstrong, AIChE J, 28, 294–301 (1982).
7. Eckert, E. R. G., and R. M. Drake, "Analysis of Heat Transfer," McGraw Hill, London (1972).
8. Einstein, A., Ann. Phys., 19, 286 (1906); Ann. Phys., 34, 591 (1911).
9. Gavis, J., and R. L. Laurence, Ind. Eng. Chem. Fund., 7, 525–527 (1968).
10. Landau, L. D., and E. M. Lifshitz, "Fluid Mechanics," Pergamon Press, Oxford, (1959).
11. Nahme, R., Ing-Archiv, 11, 191–209 (1940).
12. Pearson, J. R. A., Polym. Eng. Sci., 18, 222–229 (1978).
13. Schlichting, H., and J. Kestin, "Boundary Layer Theory," McGraw Hill, London (1955).
14. Winter, H. H., Polym. Eng. Sci., 15, 84–89 (1975).
15. Winter, H. H., Adv. Heat Transfer, 13, 205–267 (1977).

REFERENCES FOR FURTHER READING

Equation of Energy:
  Bird, Stewart and Lightfoot, 1960
  Astarita and Marrucci, 1974
  Eckert and Drake, 1972

Dimensionless Groups:
  Armstrong and Winter, 1982
  Pearson, 1978
  Schlichting and Kestin, 1955
  Eckert and Drake, 1972
  Winter, 1977

Polymer Processing:
  Winter 1977, 1975
  Pearson, 1978
  Cox and Macosko, 1974
  Gavis and Laurence, 1968
  Dinh and Armstrong, 1982

Thermal Boundary Layer:
  Eckert and Drake, 1972
  Schlichting and Kestin, 1955

Heat Transfer Coefficient for FLOW WITH VISCOUS DISSIPATION:
  Eckert and Drake, 1972
  Winter, 1977

Suspension Viscosity Defined with DISSIPATION FUNCTION:
  Einstein, 1906, 1911
  Landau and Lifshitz, 1959

STUDY PROBLEMS

1. Calculate the dimensionless temperature $(T - T_o)/T_o$ in degrees Fahrenheit, Celsius, and Kelvin. Use $T =$
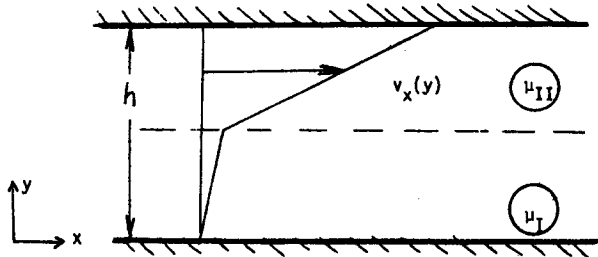
Figure 4. Shear flow of two immiscible fluids in a gap between parallel plates.

180°C and $T_o = 150°C$. Discuss how a dimensionless temperature can depend on the choice of temperature scale. HINT: Each temperature scale refers to a different temperature as zero temperature.

2. State the system of equations for the velocity, the temperature, and the viscosity for flow in a pipe with isothermal walls. How are the equations coupled with each other?

3. When is the dissipation function positive and when is it negative?

4. Consider shear flow of two immiscible Newtonian fluids in a narrow gap between two parallel plates, see Figure 4. The flow is due to the parallel movement of the upper plate. The lower half of the gap is filled with fluid I (viscosity μ_I) and the upper half with fluid II (viscosity μ_{II}). The viscosities differ by a factor of 10: $\mu_I = 10\mu_{II}$. Where is the rate of viscous dissipation higher, in the viscous fluid I or in the less viscous fluid II?

HOMEWORK PROBLEMS

1. Calculate the Nahme number for pipe flow of Example 1. Use $\Delta T_{rheol} = 50$ K and $k = 0.2$ W/mK.

2. Use slip data from the literature to determine viscous dissipation in the slip layer.
   Ref.: L. L. Blyler and A. C. Hart, "Capillary Flow Instability of Ethylene Polymer Melts," *Polym. Eng. Sci., 10,* 193–203 (1970).

3. Extend Example 4 to a linear viscoelastic material with a shear stress

$$\tau_{xy} + \lambda \frac{\partial \tau_{xy}}{\partial t} = \mu \dot{\gamma}_{xy}$$

   Hint: Determine the time dependent shear stress $\tau_{xy}(t)$ first.

4. Calculate the rate of viscous dissipation $\phi(r, \theta)$ in a Newtonian fluid which flows around a single sphere (see Reference *3*, p. 133).

Series C:
TRANSPORT

Volume 7:

Calculation and Measurement Techniques for Momentum, Energy and Mass Transfer

R. J. Gordon, Series Editor

AMERICAN INSTITUTE OF CHEMICAL ENGINEERS

INTRODUCTION

In 1975 a new venture in education by and for the chemical engineering community was initiated. Prepared by the CACHE Corporation (Computer Aids for Chemical Engineering Education) and under the sponsorship of the National Science Foundation (Grant HES 75-03911), a series of small self-study fundamental concept modules for various areas of chemical engineering were commissioned, Chemical Engineering Modular Instruction, CHEMI.

It has been found in recent studies that modular study is more effective than traditional instruction in both university and continuing education settings. This is due in large measure to the discrete focus of each module, which allows the student to tailor the speed and order of his or her study. In addition, since the modules have different authors, each writing in his or her area of special expertise, they can be produced more quickly, and students may be asured of timely information. Finally, these modules have been tested in the classroom prior to their publication.

The educational effect of modular study is to reduce, in general, the number of hours required to teach a given subject; it is expected that the decreased time and expense involved in engineering education, when aided by modular instruction, will attract a larger number of students to engineering, including those who have not traditionally chosen engineering. For the practicing engineer, the modules are intended to enhance or broaden the skills he or she has already acquired, and to make available new fields of expertise.

The modules were designed with a variety of applications in mind. They may be pursued in a number of contexts: as outside study, special projects, entire university courses (credit or non-credit), review courses, or correspondence courses; and they may be studied in a variety of modes: as supplements to course work, as independent study, in continuing education programs, and in the traditional student/teacher mode.

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